

## RESEARCH ARTICLE

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## Derivation and Solution of the Omega Equation Associated With a Balance Theory on the Sphere

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## Key Points:

- A global balance theory, similar to quasi-geostrophic theory, can be developed by decomposing the wind into nondivergent and irrotational components
- A global omega equation, similar to the quasi-geostrophic omega equation, can be derived from the global balance system
- The global omega equation may be solved over the sphere using vertical normal modes and spherical harmonics

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**Abstract** The quasi-geostrophic omega equation has been used extensively to examine the large-scale vertical velocity patterns of atmospheric systems. It is derived from the quasi-geostrophic equations, a balanced set of equations based on the partitioning of the horizontal wind into a geostrophic and an ageostrophic component. Its use is limited to higher latitudes, however, as the geostrophic balance is undefined at the equator. In order to derive an omega equation which can be used at all latitudes, a new balanced set of equations is developed. Three key steps are used in the formulation. First, the horizontal wind is decomposed into a nondivergent and an irrotational component. Second, the Coriolis parameter is assumed to be slowly varying, such that it may be moved in and out of horizontal derivative operators as necessary to simplify the derivation. Finally, the mass field is formulated from the nondivergent wind field. The resulting balanced set of equations and the omega equation derived from them take a similar form to the quasi-geostrophic equations, yet are valid over the whole sphere. A method of solution to the global omega equation using vertical normal modes and spherical harmonics is presented, along with a middle-latitude and low-latitude example.

**Plain Language Summary** Most high impact weather occurs where the atmosphere is experiencing rising motion. Vertical motions can occur on a variety of scales, and this paper explores the upward and downward currents on length scales typical of extratropical cyclones and tropical waves. A commonly-used tool for evaluating such motions is the quasi-geostrophic omega equation. The quasi-geostrophic omega equation is typically formulated using a constant Coriolis parameter, which limits the latitudinal extent of its application. In this paper, an omega equation is derived which is valid over the whole sphere. A method for solution of the new global omega equation is presented, one which employs a normal mode transform in the vertical, and a spherical harmonic transform in the horizontal. Examples of the application of the global omega equation are given for both a midlatitude cyclone as well as a tropical wave.

## 1. Introduction

Quasi-geostrophic theory has been vastly used in the atmospheric sciences. Originally derived by Charney (1948), this description of fluid motion is formulated from a scale analysis of the primitive equations that retains only those motions characterized by a small Rossby number and for which the vertical scale is small compared to the horizontal scale. In this system, the horizontal velocity is decomposed into a geostrophic component and an ageostrophic component. The ageostrophic component is assumed to be smaller in magnitude than the geostrophic component and is neglected in many, but not all, terms—thus the name *quasi*-geostrophic. This set of equations is complex enough to contain many of the detailed structures present in the atmosphere, yet simple enough that analytical solutions may be found in many of its applications.

While frequently employed in the analysis of midlatitude weather systems in an  $f$  or  $\beta$ -plane configuration, the quasi-geostrophic system has seen little use in the Tropics because the geostrophic wind becomes unrealistically large at low latitudes and is undefined altogether at the equator. Thus, quasi-geostrophic theory is frequently applied to higher latitude features such as extratropical cyclones, the propagation of planetary waves, turbulence, and the stability of jet streams, but has seen limited use in the study of low-latitude phenomena such as tropical waves and tropical cyclones. In addition, processes which occur over a wide

latitude range, such as the transition of a tropical cyclone to an extratropical cyclone are difficult to study with quasi-geostrophic theory, again owing to the inaccuracy of the geostrophic approximation at low latitudes.

In order to describe features occurring at any latitude, or over a wide range of latitudes, a global “quasi-geostrophic” theory is necessary. Although perhaps not well known, Kuo (1959) and Charney and Stern (1962) formulated such a theory over a half-century ago. Essential to this formulation is the decomposition of the horizontal wind field not into geostrophic and ageostrophic components, but rather into nondivergent and irrotational components. From there, approximations similar to those made in quasi-geostrophic theory are applied to the primitive equations to arrive at a balanced system which is valid for the whole sphere. Not surprisingly, this system of equations is very similar to the system of equations governing quasi-geostrophic motions, and applying the equations to the analysis of the atmosphere proceeds along the same path. In particular, the vorticity equation can be combined with the thermodynamic equation in one of two ways. The first approach is to eliminate the vertical motion between the two equations, resulting in a single equation governing the time tendency of the streamfunction. From this equation, the potential vorticity and the conditions of its conservation can be defined. This approach was taken by Schubert et al. (2009) and Verkley (2009) in the context of a shallow-water system. The work presented here takes the alternate approach to combining the vorticity equation and the thermodynamic equation. In this combination, the time tendency of the streamfunction is eliminated between the two equations. The result is a second-order, linear, partial differential equation describing the vertical motion field which accompanies the balanced flow—the global omega equation. This paper examines the global omega equation, including its derivation, the governing boundary conditions, and a description of its solution, which involves a normal mode transform in the vertical and a spherical harmonic transform in the horizontal. Before the omega equation is discussed, however, the system of equations from which it is derived is developed.

## 2. Balance Theory on the Sphere

### 2.1. Considerations in Extending Quasi-Geostrophic Theory to the Sphere

In order to derive an omega equation applicable to the entire sphere, a set of equations governing a balanced flow on the sphere must be developed. Quasi-geostrophic theory relies on the decomposition of the horizontal velocity into a geostrophic and an ageostrophic component. In this decomposition, there are two ways to define the geostrophic wind (Blackburn, 1985). The first is to replace the Coriolis parameter,  $f$ , with a constant,  $f_0$ , associated with a particular latitude. While such an assumption is often used in midlatitude studies, it is hardly appropriate to consider one value for the Coriolis parameter as applicable to the entire Earth. The second option is to retain the latitudinal variation in  $f$  in the definition of the geostrophic wind. This option cannot be applied to low latitudes, however, as the geostrophic balance breaks down due to the decrease in magnitude of the Coriolis parameter, resulting in a singularity at the equator where  $f = 0$ . Therefore, an approach alternate to geostrophic balance is needed.

Bluestein (1992) notes that, for large-scale midlatitude flows, the divergence can be up to an order of magnitude smaller than the relative vorticity. In his analysis, Charney (1963) reports that large-scale flows in the Tropics are also characterized by a small value of divergence. A balance theory relying on the assumption of the approximate nondivergence of the horizontal wind is therefore better suited to global motions than a theory based on geostrophic balance of the horizontal winds. This approach to obtaining a global balance theory was used by Kuo (1959) and Charney (1960). Schubert et al. (2009) and Verkley (2009) recently revisited the topic, examining the behavior of balanced, shallow-water Rossby waves on the sphere. The balance theory on the sphere will be based on the decomposition of the horizontal wind into nondivergent and irrotational components. The derivation proceeds, however, using similar scaling arguments applied by Charney (1948) and Phillips (1963) to midlatitude quasi-geostrophic theory.

### 2.2. From the Primitive Equations to the Balanced System

With pressure as the vertical coordinate, the dry, adiabatic, hydrostatic, primitive equations obeying the ideal gas law may be written as

$$\frac{D_3 \mathbf{v}}{Dt} + f \mathbf{k} \times \mathbf{v} + \nabla \Phi = \mathbf{F} \quad (1)$$

$$\frac{\partial \Phi}{\partial p} + \frac{RT}{p} = 0 \tag{2}$$

$$\nabla \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = 0 \tag{3}$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T - \frac{\Gamma}{p} \omega = 0 \tag{4}$$

$$\Gamma = \left( \frac{RT}{c_p} - p \frac{\partial T}{\partial p} \right), \tag{5}$$

where  $f = 2\Omega\mu$  is the Coriolis parameter, with  $\Omega$  the rotation rate of the Earth and  $\mu$  the sine of the latitude.  $\frac{D_3}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \omega \frac{\partial}{\partial p}$  is the total derivative.

The above equations describe the relationship among the dependent variables  $\mathbf{v}$ ,  $\omega$ ,  $\Phi$ ,  $T$ , and  $\Gamma$ , which are all functions of the independent variables  $\mathbf{r}$ ,  $p$ , and  $t$ , where  $\mathbf{r} = (\lambda\mathbf{i}, \mu\mathbf{j})$  is a horizontal position vector with longitude  $\lambda$ ,  $p$  is pressure, and  $t$  is time. A vector representing the effects of boundary layer friction,  $\mathbf{F}$ , has been included in the horizontal momentum equation. In midlatitude studies, this term is often neglected because of the strength of the quasi-geostrophic forcing. In the Tropics, however, the quasi-geostrophic effects are much weaker, even to the point of being equaled or surpassed in magnitude by the frictional contribution (Baumhefner, 1968; Krishnamurti, 1968). The frictional term has therefore been retained in the equations of motion.

In order to derive the equations governing a balance theory on the sphere, approximations to the primitive equations will be made which isolate the features characterized by a vertical scale which is small compared to the horizontal scale and having a small Rossby number. In particular, the momentum equations and the thermodynamic equation will be modified.

Before applying the necessary approximations to the momentum equation (1) to arrive at the balanced system, it is convenient to replace it with equations governing the divergence and the vertical component of the vorticity of the flow. These equations are derived, respectively, by taking  $\nabla \cdot$  and  $\mathbf{k} \cdot \nabla \times$  of the momentum equation (1):

$$\frac{\partial \delta}{\partial t} + \nabla^2 \left( \Phi + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \zeta(\zeta + f) + \mathbf{v} \cdot \nabla(\zeta + f) \times \mathbf{k} + \omega \frac{\partial \delta}{\partial p} + \nabla \omega \cdot \frac{\partial \mathbf{v}}{\partial p} = \nabla \cdot \mathbf{F}$$

and

$$\frac{D_3(\zeta + f)}{Dt} - (\zeta + f) \frac{\partial \omega}{\partial p} + \mathbf{k} \cdot \left( \nabla \omega \times \frac{\partial \mathbf{v}}{\partial p} \right) = \mathbf{k} \cdot \nabla \times \mathbf{F},$$

where  $\delta = \nabla \cdot \mathbf{v}$  is the divergence of the horizontal wind field, and  $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v}$  is the vertical component of the isobaric relative vorticity.

Decomposing the horizontal wind field into nondivergent and irrotational components allows the horizontal wind vector to be written as  $\mathbf{v} = \mathbf{v}_\psi + \mathbf{v}_\chi$ . The nondivergent wind is given by  $\mathbf{v}_\psi = \mathbf{k} \times \nabla \psi$ , where  $\psi$  is the streamfunction, and the irrotational wind is given by  $\mathbf{v}_\chi = \nabla \chi$ , where  $\chi$  is the velocity potential. The vertical component of the relative vorticity is then  $\zeta = \nabla^2 \psi$ , and the divergence of the horizontal wind is given by  $\delta = \nabla^2 \chi$ , with  $\nabla \cdot \mathbf{v}_\psi = 0$  and  $\nabla \times \mathbf{v}_\chi = 0$ .

The approximate form of the divergence equation ultimately defines the balance assumed by the system—nonlinear balance, linear balance, or geostrophic balance. By neglecting in the divergence equation terms involving the irrotational wind component, the term  $\nabla \omega \cdot \frac{\partial \mathbf{v}}{\partial p}$ , the friction term, and the nonlinear terms, the balance theory on the sphere begins with the linear balance:

$$\nabla^2 \Phi - \nabla \cdot (f \nabla \psi) = 0. \tag{6}$$

By considering  $f$  to be a slowly varying function of latitude in comparison to  $\psi$ , it can be moved into the gradient operator in (6). Thus, the balance equation can be written as  $\nabla^2(\Phi - f\psi) = 0$ . The solution to this equation on a sphere is  $\Phi - f\psi = C$ , where  $C$  is a constant. This equation holds for each isobaric surface, but as will

be discussed subsequently,  $C$  must be set to the base-state geopotential of the isobaric surface. The final version of the balance equation becomes

$$f\psi(\lambda, \mu, p) = \Phi(\lambda, \mu, p) - \bar{\Phi}(p), \tag{7}$$

where  $\bar{\Phi}(p)$  is the base-state geopotential which is related to the base-state temperature profile through the hydrostatic equation. Thus, the Coriolis parameter maintains its full latitudinal variation, but the balance condition will not be valid for motions with large meridional extent. In comparing the Rossby-Haurwitz wave frequencies of the shallow-water version of the global balance theory with those derived from the primitive equations, Schubert et al. (2009) found excellent agreement in all cases but the sectoral harmonics, i.e., harmonics with low meridional wavenumber.

Equation (7) describes merely the relationship between the temperature, or mass, field ( $\Phi$ ) and the wind, or momentum, field ( $\psi$ ). It does not require that either the mass field or the momentum field be the independent variable. Phillips (1958) explored the two options in the context of numerical weather prediction, using a midlatitude cyclone over the eastern United States as an example. He came to the conclusion that it is more important to have an estimate of the wind field, and from that derive the associated temperature field than it is to have an estimate of the temperature field, and from that derive the wind field. More recent work on this idea of “slaving” and “slaved” variables can be found in Warn et al. (1995), Mohebalhojeh (2002), and Nielsen-Gammon and Gold (2008).

Although Phillips’ domain covered only a portion of the midlatitudes, one might consider a verification of his conclusion to be borne out for the global balance theory by examining (7). If the balance is constructed by determining the streamfunction from the geopotential field, then  $\psi = \frac{\Phi(\lambda, \mu, p) - \bar{\Phi}(p)}{f}$ , which is indeterminate or singular at the equator, depending on how  $\bar{\Phi}(p)$  is defined. If, on the other hand, the balance is constructed by determining the geopotential field from the streamfunction, no such problem exists. It does however require that, along the equator, the geopotential field is equal to the base-state geopotential for each pressure level, so that the geopotential perturbations are identically zero.

The vorticity equation will be approximated by neglecting the horizontal advection of absolute vorticity by the irrotational component of the wind, the vertical advection of absolute vorticity, the stretching of the relative vorticity, and the twisting term (Kuo, 1959). The new vorticity equation is then given by

$$\frac{D(\nabla^2\psi + f)}{Dt} - f \frac{\partial\omega}{\partial p} = \mathbf{k} \cdot \nabla \times \mathbf{F},$$

where  $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_\psi \cdot \nabla$  is the total derivative following the nondivergent component of the balanced wind. This form of the vorticity equation is consistent with Cases I and II of Charney (1960), only he includes the advection of the planetary vorticity by the irrotational wind and neglects friction.

The thermodynamic equation will also be approximated following the arguments listed in Kuo (1959). The horizontal advection of  $T$  by the irrotational component of the flow is neglected, and the static stability  $\Gamma$  is replaced with a base-state static stability that is a function of pressure only  $\bar{\Gamma} = \left(\frac{R\bar{T}}{c_p} - p \frac{d\bar{T}}{dp}\right)$ . With these changes the thermodynamic equation becomes

$$\frac{DT}{Dt} - \frac{\bar{\Gamma}}{p} \omega = 0. \tag{8}$$

In formulating the omega equation, it is convenient to replace the temperature in the total derivative with an expression involving the streamfunction. This is accomplished through the use of the hydrostatic equation along with the balance condition. The thermodynamic equation becomes:

$$\frac{D\left(f \frac{\partial\psi}{\partial p}\right)}{Dt} + \frac{R\bar{\Gamma}}{p^2} \omega = 0.$$

### 2.3. The Balanced System of Equations

With the approximations to the primitive equations made, and with  $\nabla^2\chi$  replacing  $\nabla \cdot \mathbf{v}$  in the continuity equation, the balanced theory on the sphere may be presented:

**Global Balance Theory**

$$f\psi = \Phi - \bar{\Phi} \tag{9}$$

$$\frac{D(\nabla^2\psi + f)}{Dt} - f \frac{\partial\omega}{\partial p} = \mathbf{k} \cdot \nabla \times \mathbf{F} \tag{10}$$

$$\frac{\partial\Phi}{\partial p} + \frac{RT}{p} = 0 \tag{11}$$

$$\nabla^2\chi + \frac{\partial\omega}{\partial p} = 0 \tag{12}$$

$$\frac{D\left(f \frac{\partial\psi}{\partial p}\right)}{Dt} + \frac{R\bar{\Gamma}}{p^2}\omega = 0 \tag{13}$$

$$\mathbf{v}_\psi = \mathbf{k} \times \nabla\psi. \tag{14}$$

**2.4. The Thermal Wind Relationship**

The derivation of the associated thermal wind relation, which will be used in the construction of the global omega equation, begins with the balance condition

$$f\psi = \Phi - \bar{\Phi}.$$

Operating with  $\partial/\partial p$  and using  $\partial\Phi/\partial p = -\alpha$  and  $d\bar{\Phi}/dp = -\bar{\alpha}$  gives

$$f \frac{\partial\psi}{\partial p} = -(\alpha - \bar{\alpha}). \tag{15}$$

Applying the assumption of a slowly varying  $f$ , and operating with  $\mathbf{k} \times \nabla$  on the above equation results in the thermal wind equation for the balance theory on the sphere:

$$f \frac{\partial\mathbf{v}_\psi}{\partial p} = -\mathbf{k} \times \nabla\alpha. \tag{16}$$

**3. The Omega Equation**

**3.1. Derivation**

The omega equation is a diagnostic equation describing the large-scale vertical motion of the atmosphere, and is derived by eliminating the streamfunction tendency term from the vorticity and thermodynamic equations. As with the midlatitude omega equation, the global omega equation can be written in two ways, depending on the form of the forcing. The first, or traditional, form has the forcing divided into a vorticity component and a thermal component. In the second form, the forcing is written in terms of the so-called **Q** vector.

In order to eliminate the streamfunction tendency from the balanced vorticity equation and the balanced thermodynamic equation, (10) is operated on with  $\frac{\partial}{\partial p}$  after multiplying by  $-f$ , then added to  $\nabla^2$  of (13). The result is

$$\begin{aligned} \frac{R\bar{\Gamma}}{p^2}\nabla^2\omega + f^2\frac{\partial^2\omega}{\partial p^2} &= \frac{\partial}{\partial p} [f\mathbf{v}_\psi \cdot \nabla(\zeta + f)] - \nabla^2 \left[ f\mathbf{v}_\psi \cdot \nabla \left( \frac{\partial\psi}{\partial p} \right) \right] \\ &- \frac{\partial}{\partial p} (f\mathbf{k} \cdot \nabla \times \mathbf{F}). \end{aligned} \tag{17}$$

If friction is neglected and  $\psi_g = \frac{\Phi}{f_0}$  is the geostrophic streamfunction, equation (17) is analogous to the frictionless, midlatitude quasi-geostrophic omega equation:

$$\frac{R\bar{\Gamma}}{p^2} \nabla^2 \omega + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = \frac{\partial}{\partial p} [f_0 \mathbf{v}_g \cdot \nabla (\zeta_g + f)] - \nabla^2 \left[ f_0 \mathbf{v}_g \cdot \nabla \left( \frac{\partial \psi_g}{\partial p} \right) \right]. \quad (18)$$

In comparing (17) with (18), it can be seen that the forms of the two omega equations are identical—with a 3-D, Laplacian-type operator on the left-hand side, forced on the right-hand side by two terms (when friction is neglected). The first is the vertical derivative of the advection of the absolute vorticity. The second is the Laplacian of the advection of the vertical derivative of the streamfunction. By the hydrostatic equation, this derivative is proportional to the temperature. Large-scale vertical motions on the sphere, then, are forced by the same mechanisms by which large-scale vertical motions at midlatitudes are forced—by differential cyclonic vorticity advection and the Laplacian of the thermal advection.

In addition to the similarities, two differences can be seen. The advectations are accomplished in (18) by the geostrophic wind, but by the nondivergent wind in (17). In the midlatitude form, the Coriolis parameter assumes a constant value,  $f_0$ , except in the vorticity advection term. In the global omega equation, the Coriolis parameter retains its full variability. The complete range of values, however, includes  $f = 0$  at the equator, in which case (17) indicates that the forcing vanishes.

Similar to the quasi-geostrophic omega equation, the forcing of the global omega equation may also be written in the form of a vector,  $\mathbf{Q}$ . Using the slowly varying  $f$  approximation, the first two terms on the right-hand side of (17) can be written as:

$$f \left\{ \frac{\partial}{\partial p} [\mathbf{v}_\psi \cdot \nabla (\zeta + f)] - \nabla^2 \left[ \mathbf{v}_\psi \cdot \nabla \left( \frac{\partial \psi}{\partial p} \right) \right] \right\}.$$

Since  $\mathbf{v}_\psi$  is nondivergent, the advection by  $\mathbf{v}_\psi$  of the absolute vorticity is equal to the divergence of the vorticity flux, that is,  $\mathbf{v}_\psi \cdot \nabla (\zeta + f) = \nabla \cdot [\mathbf{v}_\psi (\zeta + f)]$ . Making this substitution and manipulating the vertical derivative, the inner product, and gradient operators, the forcing term becomes

$$f \nabla \cdot \left\{ \frac{\partial}{\partial p} [\mathbf{v}_\psi (\zeta + f)] - \nabla \left[ \mathbf{v}_\psi \cdot \frac{\partial}{\partial p} (\nabla \psi) \right] \right\}.$$

The term inside the second set of brackets can be rewritten as  $\mathbf{v}_\psi \cdot \frac{\partial}{\partial p} (\nabla \psi) = \frac{\partial}{\partial p} (\mathbf{v}_\psi \cdot \nabla \psi) - \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \psi$ . Because  $\mathbf{v}_\psi = \mathbf{k} \times \nabla \psi$  is perpendicular to  $\nabla \psi$ ,  $\mathbf{v}_\psi \cdot \nabla \psi = 0$ , from which it follows that  $\mathbf{v}_\psi \cdot \frac{\partial}{\partial p} (\nabla \psi) = -\frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \psi$ . The forcing term for the omega equation can then be written as:

$$f \nabla \cdot \left\{ \frac{\partial}{\partial p} [\mathbf{v}_\psi (\zeta + f)] + \nabla \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \psi \right) \right\}.$$

Using the identity  $\nabla (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$  on the second term and noting that  $\nabla \psi \times (\nabla \times \frac{\partial \mathbf{v}_\psi}{\partial p}) = \nabla \psi \times \frac{\partial \zeta}{\partial p} \mathbf{k} = -\frac{\partial \zeta}{\partial p} \mathbf{v}_\psi$  gives

$$f \nabla \cdot \left\{ \frac{\partial}{\partial p} [\mathbf{v}_\psi (\zeta + f)] + \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \right) \nabla \psi + (\nabla \psi \cdot \nabla) \frac{\partial \mathbf{v}_\psi}{\partial p} - \frac{\partial \zeta}{\partial p} \mathbf{v}_\psi \right\}.$$

Expanding the first term and adding  $(\frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla) \nabla \psi - (\frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla) \nabla \psi = 0$  gives

$$f \nabla \cdot \left[ 2 \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \right) \nabla \psi + f \frac{\partial \mathbf{v}_\psi}{\partial p} + \zeta \frac{\partial \mathbf{v}_\psi}{\partial p} + (\nabla \psi \cdot \nabla) \frac{\partial \mathbf{v}_\psi}{\partial p} - \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \right) \nabla \psi \right].$$

Using the identity  $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}$ , the last three terms can be combined to give  $\nabla \times \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \times \nabla \psi \right)$ . Using this in the forcing term gives

$$f \nabla \cdot \left[ 2 \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \right) \nabla \psi + f \frac{\partial \mathbf{v}_\psi}{\partial p} + \nabla \times \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \times \nabla \psi \right) \right].$$

Using the result  $\nabla \cdot \left[ \nabla \times \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \times \nabla \psi \right) \right] = 0$ , the forcing takes the form

$$f \nabla \cdot \left[ 2 \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \right) \nabla \psi + f \frac{\partial \mathbf{v}_\psi}{\partial p} \right],$$

or

$$f \nabla \cdot \left[ 2 \left( \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \right) \nabla \psi \right] + f \nabla \cdot \left( f \frac{\partial \mathbf{v}_\psi}{\partial p} \right).$$

Up to this point, the derivation has been independent of any coordinate system. To continue the derivation, however, spherical coordinates will be assumed. Expanding  $\nabla \psi$  in spherical coordinates and using the assumption of a slowly varying  $f$ , the first term of the above equation may be written as

$$2 \nabla \cdot \left[ \left( f \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \right) (v_\psi \mathbf{i} - u_\psi \mathbf{j}) \right]$$

or equivalently,

$$-2 \nabla \cdot \left[ - \left( f \frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla \right) (v_\psi \mathbf{i} - u_\psi \mathbf{j}) \right].$$

If the product  $\frac{\partial \mathbf{v}_\psi}{\partial p} \cdot \nabla$  is expanded and operates on  $v_\psi \mathbf{i} - u_\psi \mathbf{j}$ , the result is

$$\begin{aligned} \mathbf{Q} = (Q_1, Q_2, Q_3) = & -f \left[ \left( \frac{\partial u_\psi}{\partial p} \right) \left( \frac{\partial v_\psi}{\partial \lambda \cos \phi} + \frac{u_\psi \tan \phi}{a} \right) + \left( \frac{\partial v_\psi}{\partial p} \right) \left( \frac{\partial v_\psi}{\partial \phi} \right) \right] \mathbf{i} \\ & -f \left[ \left( \frac{\partial u_\psi}{\partial p} \right) \left( -\frac{\partial u_\psi}{\partial \lambda \cos \phi} + \frac{v_\psi \tan \phi}{a} \right) - \left( \frac{\partial v_\psi}{\partial p} \right) \left( \frac{\partial u_\psi}{\partial \phi} \right) \right] \mathbf{j} \\ & -f \left[ - \left( \frac{\partial u_\psi}{\partial p} \right) \left( \frac{v_\psi}{a} \right) + \left( \frac{\partial v_\psi}{\partial p} \right) \left( \frac{u_\psi}{a} \right) \right] \mathbf{k}. \end{aligned} \tag{19}$$

Here the variation with latitude and longitude of the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$  has been taken into account, that is,  $\frac{\partial \mathbf{i}}{\partial \lambda \cos \phi} = \frac{\tan \phi}{a} \mathbf{j} - \frac{1}{a} \mathbf{k}$ ,  $\frac{\partial \mathbf{i}}{\partial \phi} = 0$ ,  $\frac{\partial \mathbf{j}}{\partial \lambda \cos \phi} = -\frac{\tan \phi}{a} \mathbf{i}$ , and  $\frac{\partial \mathbf{j}}{\partial \phi} = -\frac{1}{a} \mathbf{k}$ , where  $a$  is the radius of the Earth (Holton, 1992). In addition, because  $\nabla$  has been defined to be a horizontal operator, equation (19) implies that the component of  $\mathbf{Q}$  in the  $\mathbf{k}$  direction may be removed from further consideration.

$\mathbf{Q}$  is a generalized form of that discussed by Hoskins et al. (1978). It can also be written using only horizontal derivative operators through the use of the components of the thermal wind equation  $f \frac{\partial u_\psi}{\partial p} = \frac{\partial z}{\partial \phi}$  and  $f \frac{\partial v_\psi}{\partial p} = -\frac{\partial z}{\partial \lambda \cos \phi}$ . Using these relations in (19) gives

$$Q_1 = - \left[ \left( \frac{\partial z}{\partial \phi} \right) \left( \frac{\partial v_\psi}{\partial \lambda \cos \phi} + \frac{u_\psi \tan \phi}{a} \right) - \left( \frac{\partial z}{\partial \lambda \cos \phi} \right) \left( \frac{\partial v_\psi}{\partial \phi} \right) \right] \tag{20}$$

and

$$Q_2 = - \left[ \left( \frac{\partial z}{\partial \phi} \right) \left( -\frac{\partial u_\psi}{\partial \lambda \cos \phi} + \frac{v_\psi \tan \phi}{a} \right) + \left( \frac{\partial z}{\partial \lambda \cos \phi} \right) \left( \frac{\partial u_\psi}{\partial \phi} \right) \right]. \tag{21}$$

$Q_1$  and  $Q_2$  may now be computed using variables at only one level. The terms  $\frac{u_\psi \tan \phi}{a}$  and  $\frac{v_\psi \tan \phi}{a}$ , however, present a difficulty as  $\tan \phi$  is unbounded at the poles. To avoid singularities at the poles, the definition of vorticity and divergence will be used to eliminate these terms in the definitions of  $Q_1$  and  $Q_2$ , respectively.

The equation for the vertical component of the relative vorticity may be written as

$$\zeta = \frac{\partial v_\psi}{a \cos \phi \partial \lambda} - \frac{\partial (u_\psi \cos \phi)}{a \cos \phi \partial \phi} = \frac{\partial v_\psi}{a \cos \phi \partial \lambda} - \frac{\partial u_\psi}{a \partial \phi} + \frac{u_\psi \tan \phi}{a}.$$

Using this result in (20) gives

$$Q_1 = - \left[ \left( \frac{\partial z}{\partial \phi} \right) \left( \frac{\partial u_\psi}{\partial \phi} + \zeta \right) - \left( \frac{\partial z}{\partial \lambda \cos \phi} \right) \left( \frac{\partial v_\psi}{\partial \phi} \right) \right]. \tag{22}$$

The 2-D divergence of  $\mathbf{v}_\psi$  is

$$\delta = \frac{\partial u_\psi}{a \cos \phi \partial \lambda} + \frac{\partial (v_\psi \cos \phi)}{a \cos \phi \partial \phi} = 0.$$

Expanding the second term, the above equation becomes

$$0 = \frac{\partial u_\psi}{a \cos \phi \partial \lambda} + \frac{\partial v_\psi}{a \partial \phi} - \frac{v_\psi \tan \phi}{a}.$$

Using this result in (21) gives

$$Q_2 = - \left[ \left( \frac{\partial \alpha}{a \partial \phi} \right) \left( \frac{\partial v_\psi}{a \partial \phi} \right) + \left( \frac{\partial \alpha}{a \cos \phi \partial \lambda} \right) \left( \frac{\partial u_\psi}{a \partial \phi} \right) \right]. \quad (23)$$

The  $\mathbf{Q}$  form of the omega equation can now be written as

$$\frac{R\bar{\Gamma}}{p^2} \nabla^2 \omega + f^2 \frac{\partial^2 \omega}{\partial p^2} = -2\nabla \cdot \mathbf{Q} + f \nabla \cdot \left( f \frac{\partial \mathbf{v}_\psi}{\partial p} \right).$$

The second term is typically of the same order of magnitude as  $-2\nabla \cdot \mathbf{Q}$ , and so will be retained (Blackburn, 1985). This term may also be written in terms of variables on a single level through the use of the thermal wind equation (16):

$$f \nabla \cdot \left( f \frac{\partial \mathbf{v}_\psi}{\partial p} \right) = f \nabla \cdot (-\mathbf{k} \times \nabla \alpha).$$

Returning friction to the forcing, the  $\mathbf{Q}$  form of the omega equation can then be written as

$$\frac{R\bar{\Gamma}}{p^2} \nabla^2 \omega + f^2 \frac{\partial^2 \omega}{\partial p^2} = -2\nabla \cdot \mathbf{Q} - f \nabla \cdot (\mathbf{k} \times \nabla \alpha) - \frac{\partial}{\partial p} (f \mathbf{k} \cdot \nabla \times \mathbf{F}). \quad (24)$$

### 3.2. Boundary Conditions

In order to solve a partial differential equation, the boundary conditions must be stated. To solve the omega equation (17) or (24), the atmosphere is considered to be bounded by two isobaric surfaces,  $p = p_B$  and  $p = p_T$ . At  $p = p_T$ ,  $\omega$  is chosen to be zero, meaning that air parcels do not cross the upper boundary. At the lower boundary, it is assumed that parcels do not cross physical height surfaces, so,  $\frac{D_3 z}{Dt}$  is chosen to be zero at  $p = p_B$ . In practice, the condition on the lower boundary will be written in terms of the geopotential  $\Phi = gz$ , so that  $g \frac{D_3 z}{Dt} = \frac{D_3 \Phi}{Dt} = 0$ , where  $g$  is the acceleration due to gravity, which is assumed to be constant. Writing the geopotential in terms of the balance condition (9), as well as expanding the material derivative gives

$$\omega \frac{d\bar{\Phi}}{dp} + \frac{\partial(f\psi)}{\partial t} + \mathbf{v}_\psi \cdot \nabla(f\psi) = 0, \quad \text{at } p = p_B,$$

where the horizontal advection of  $\bar{\Phi}$  is zero because  $\bar{\Phi} = \bar{\Phi}(p)$ , the horizontal advection of  $f\psi$  is accomplished by  $\mathbf{v}_\psi$  only, and the vertical advection of  $f\psi$  has been neglected. These approximations are consistent with the approximations used to develop the balance theory on the sphere. Using the approximation of slowly varying  $f$ , the third term of the above equation can be rewritten as  $\mathbf{v}_\psi \cdot f \nabla \psi$ . Because  $\mathbf{v}_\psi = \mathbf{k} \times \nabla \psi$  is perpendicular to  $\nabla \psi$ ,  $\mathbf{v}_\psi \cdot f \nabla \psi = 0$ . The lower boundary condition becomes

$$\omega \frac{d\bar{\Phi}}{dp} + \frac{\partial(f\psi)}{\partial t} = 0. \quad (25)$$

In order to apply the lower boundary condition, it will be convenient to rewrite it using the balanced vorticity equation (10). This equation applies at all levels, including the lower boundary, and can be written as

$$\frac{\partial \zeta}{\partial t} + \mathbf{v}_\psi \cdot \nabla(\zeta + f) - f \frac{\partial \omega}{\partial p} - \mathbf{k} \cdot \nabla \times \mathbf{F} = 0. \quad (26)$$

Multiplying (26) by  $-f$ , adding it to  $\nabla^2$  of (25) and recognizing that  $\frac{d\bar{\Phi}}{dp} = -\frac{R\bar{\Gamma}}{p}$ , the lower boundary condition can be written as

$$f^2 \frac{\partial \omega}{\partial p} - \frac{R\bar{\Gamma}}{p} \nabla^2 \omega = f \mathbf{v}_\psi \cdot \nabla(\zeta + f) - f \mathbf{k} \cdot \nabla \times \mathbf{F},$$

or by noting that  $\mathbf{v}_\psi$  is nondivergent,

$$f^2 \frac{\partial \omega}{\partial p} - \frac{R\bar{\Gamma}}{p} \nabla^2 \omega = f \nabla \cdot [\mathbf{v}_\psi (\zeta + f)] - f \mathbf{k} \cdot \nabla \times \mathbf{F}.$$

The global omega equation and its boundary conditions have now been defined and are repeated here.

**Global Omega Equation and Boundary Conditions**

$$\frac{R\bar{\Gamma}}{p^2} \nabla^2 \omega + f^2 \frac{\partial^2 \omega}{\partial p^2} = \frac{\partial}{\partial p} [f \mathbf{v}_\psi \cdot \nabla (\zeta + f)] - \nabla^2 \left[ f \mathbf{v}_\psi \cdot \nabla \left( \frac{\partial \psi}{\partial p} \right) \right] - \frac{\partial}{\partial p} (f \mathbf{k} \cdot \nabla \times \mathbf{F})$$

or

$$\frac{R\bar{\Gamma}}{p^2} \nabla^2 \omega + f^2 \frac{\partial^2 \omega}{\partial p^2} = -2 \nabla \cdot \mathbf{Q} - f \nabla \cdot (\mathbf{k} \times \nabla \alpha) - \frac{\partial}{\partial p} (f \mathbf{k} \cdot \nabla \times \mathbf{F}) \tag{27}$$

$$\omega = 0 \text{ at } p = p_T \tag{28a}$$

$$f^2 \frac{\partial \omega}{\partial p} - \frac{R\bar{\Gamma}}{p} \nabla^2 \omega = f \nabla \cdot [\mathbf{v}_\psi (\zeta + f)] - f \mathbf{k} \cdot \nabla \times \mathbf{F} \text{ at } p = p_B \tag{28b}$$

**3.3. Solution**

Equation (27) with boundary conditions (28a) and (28b) will be solved using transform techniques—a normal mode transform in the vertical and a spherical harmonic transform in the horizontal. As this equation will be solved on the sphere, only vertical boundary conditions are needed, as there are no lateral boundaries.

**3.3.1. Vertical Normal Mode Transform**

A vertical normal mode transform is used to separate the vertical structure of the atmosphere from the horizontal structure. The procedure used here follows that developed by Fulton and Schubert (1985), although they performed the transform on equations involving  $u$ ,  $v$ , and  $\Phi$ . The forward and inverse transforms on  $\omega$  are, respectively,

$$\omega_\ell(\lambda, \mu) = \int_{p_T}^{p_B} \omega(\lambda, \mu, p) \frac{dV_\ell}{dp} dp + \left[ \omega(\lambda, \mu, p) \frac{p\bar{\Gamma}}{\Gamma} \frac{dV_\ell}{dp} \right]_{p_B}, \tag{29a}$$

and

$$\omega(\lambda, \mu, p) = \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) \frac{c_\ell^2 p^2}{R\bar{\Gamma}(p_B - p_T)} \frac{dV_\ell}{dp}. \tag{29b}$$

These two equations are derived in the Appendix A. The vertical structure function  $V_\ell = V_\ell(p)$  is determined by solving the Sturm-Liouville equation

$$\frac{d}{dp} \left( \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right) + \frac{V_\ell}{c_\ell^2} = 0, \tag{30}$$

with boundary conditions

$$\frac{dV_\ell}{dp} = 0 \text{ at } p = p_T \tag{31}$$

and

$$p \frac{dV_\ell}{dp} + \frac{\bar{\Gamma}}{\Gamma} V_\ell = 0 \text{ at } p = p_B, \tag{32}$$

where  $c_\ell^{-2}$  is the eigenvalue. The vertical structure functions obey the orthonormality condition

$$\frac{1}{p_B - p_T} \int_{p_T}^{p_B} V_\ell V_{\ell'} dp = \begin{cases} 1 & \text{if } \ell' = \ell \\ 0 & \text{if } \ell' \neq \ell. \end{cases} \tag{33}$$

The kernel of the vertical normal mode transforms for  $\omega$ , however, is  $\frac{dV_\ell}{dp}$ , which also obeys an orthonormality condition, namely:

$$\frac{c_\ell^2}{p_B - p_T} \left[ \int_{p_T}^{p_B} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} dp + \left( \frac{p^3 \bar{T}}{R\bar{\Gamma}^2} \frac{dV_\ell}{dp} \right)_{p_B} \right] = \begin{cases} 1 & \text{if } \ell' = \ell \\ 0 & \text{if } \ell' \neq \ell. \end{cases} \quad (34)$$

The vertical transform of the global omega equation begins by multiplying (27) by  $\frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp}$  and integrating from  $p_T$  to  $p_B$ :

$$\int_{p_T}^{p_B} \nabla^2 \omega \frac{dV_\ell}{dp} dp + f^2 \int_{p_T}^{p_B} \frac{\partial^2 \omega}{\partial p^2} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} dp = \int_{p_T}^{p_B} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} F dp, \quad (35)$$

where  $F = F(\lambda, \mu, p)$  is one of the forms of the forcing (rhs) of (27). Evaluate each term on the left-hand side, beginning with the first term

$$\int_{p_T}^{p_B} \nabla^2 \omega \frac{dV_\ell}{dp} dp.$$

If  $\nabla^2 \left( \omega \frac{p\bar{T}}{\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} - \nabla^2 \left( \omega \frac{p\bar{T}}{\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_T} = 0$  is added, the result is

$$\nabla^2 \left[ \int_{p_T}^{p_B} \omega \frac{dV_\ell}{dp} dp + \left( \omega \frac{p\bar{T}}{\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} \right] - \nabla^2 \left( \omega \frac{p\bar{T}}{\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_T}.$$

Use (29a) and the fact that in the second term only  $\omega$  is dependent on  $\lambda$  and  $\phi$  to get

$$\nabla^2 \omega_\ell - \left( \frac{dV_\ell}{dp} \frac{p\bar{T}}{\bar{\Gamma}} \nabla^2 \omega \right)_{p_B}. \quad (36)$$

Using integration by parts, the second term on the left-hand side of (35) becomes

$$f^2 \left[ \int_{p_T}^{p_B} \frac{\partial}{\partial p} \left( \frac{\partial \omega}{\partial p} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right) dp - \int_{p_T}^{p_B} \frac{\partial \omega}{\partial p} \frac{d}{dp} \left( \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right) dp \right].$$

Evaluating the first integral and using (30) on the second gives

$$f^2 \left( \left[ \frac{\partial \omega}{\partial p} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right]_{p_T} + \int_{p_T}^{p_B} \frac{\partial \omega}{\partial p} \frac{dV_\ell}{c_\ell^2} dp \right).$$

Using the boundary condition (31) on the first term and integrating by parts on the second term gives

$$\begin{aligned} & f^2 \left[ \left( \frac{\partial \omega}{\partial p} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} + \int_{p_T}^{p_B} \frac{\partial}{\partial p} \left( \omega \frac{V_\ell}{c_\ell^2} \right) dp - \int_{p_T}^{p_B} \omega \frac{d}{dp} \left( \frac{V_\ell}{c_\ell^2} \right) dp \right] \\ & = f^2 \left[ \left( \frac{\partial \omega}{\partial p} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} + \left( \omega \frac{V_\ell}{c_\ell^2} \right)_{p_T} - \int_{p_T}^{p_B} \omega \frac{1}{c_\ell^2} \frac{dV_\ell}{dp} dp \right]. \end{aligned} \quad (37)$$

Using the boundary conditions (28a) and (32) gives

$$\begin{aligned} & f^2 \left[ \left( \frac{\partial \omega}{\partial p} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} - \left( \omega \frac{p\bar{T}}{\bar{\Gamma}} \frac{1}{c_\ell^2} \frac{dV_\ell}{dp} \right)_{p_B} - \frac{1}{c_\ell^2} \int_{p_T}^{p_B} \omega \frac{dV_\ell}{dp} dp \right] \\ & = f^2 \left\{ \left( \frac{\partial \omega}{\partial p} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} - \frac{1}{c_\ell^2} \left[ \int_{p_T}^{p_B} \omega \frac{dV_\ell}{dp} dp + \left( \omega \frac{p\bar{T}}{\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} \right] \right\}. \end{aligned} \quad (38)$$

Finally, use (29a) to get

$$f^2 \left( \frac{\partial \omega}{\partial p} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} - f^2 \frac{\omega_\ell}{c_\ell^2}. \quad (39)$$

Combining (36) and (39) with the right-hand side of (35) gives

$$\nabla^2 \omega_\ell - f^2 \frac{\omega_\ell}{c_\ell^2} + f^2 \left( \frac{\partial \omega}{\partial p} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} \right)_{p_B} - \left( \frac{dV_\ell}{dp} \frac{p\bar{\Gamma}}{\bar{\Gamma}} \nabla^2 \omega \right)_{p_B} = \int_{p_T}^{p_B} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} F dp. \tag{40}$$

Using the lower boundary condition  $f^2 \frac{\partial \omega}{\partial p} - \frac{R\bar{\Gamma}}{p} \nabla^2 \omega = G(\lambda, \mu, p)$  at  $p = p_B$  (28b) and rearranging gives

$$\nabla^2 \omega_\ell - \frac{f^2}{c_\ell^2} \omega_\ell = \int_{p_T}^{p_B} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} F dp - \left( \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} G \right)_{p_B}.$$

If the Laplacian operator is expanded in terms of  $\lambda$  and  $\mu$  and the right-hand side is multiplied by  $1 = \left( \frac{p_B - p_T}{c_\ell} \right)^2 \left( \frac{c_\ell}{p_B - p_T} \right)^2$ , the result is

$$\frac{\partial^2 \omega_\ell}{(1 - \mu^2) \partial \lambda^2} + \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial \omega_\ell}{\partial \mu} \right] - \epsilon_\ell \mu^2 \omega_\ell = \left[ \frac{a(p_B - p_T)}{c_\ell} \right]^2 F_\ell(\lambda, \mu), \tag{41}$$

where  $\epsilon_\ell = 4\Omega^2 a^2 / c_\ell^2$  is Lamb's parameter and

$$F_\ell(\lambda, \mu) = \left( \frac{c_\ell}{p_B - p_T} \right)^2 \left[ \int_{p_T}^{p_B} \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} F dp - \left( \frac{p^2}{R\bar{\Gamma}} \frac{dV_\ell}{dp} G \right)_{p_B} \right].$$

In the next section, the spherical harmonic transform will be used to convert the partial differential equation (41) into an algebraic system.

### 3.3.2. Spherical Harmonic Transform

The spherical harmonic transform consists of a Fourier transform in  $\lambda$  and an associated Legendre transform in  $\mu$ . Because of the variety in notations and normalizations used in the literature in conjunction with spherical harmonics, before proceeding with the derivation of the omega equation, the notations and normalizations used in this paper will be presented. The spherical harmonic transform pair will be given by

$$\omega_{\ell mn} = \int_{-1}^1 \int_0^{2\pi} \omega_\ell(\lambda, \mu) [Y_n^m(\lambda, \mu)]^* d\lambda d\mu$$

and

$$\omega_\ell(\lambda, \mu) = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \omega_{\ell mn} Y_n^m(\lambda, \mu),$$

where \* denotes complex conjugation. The spherical harmonics,  $Y_n^m(\lambda, \mu)$ , are given by

$$Y_n^m(\lambda, \mu) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \bar{P}_n^m(\mu) \Phi_m(\lambda),$$

where  $\Phi_m(\lambda) = e^{im\lambda}$  is the Fourier component, and  $\bar{P}_n^m(\mu) = (1 - \mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} \left\{ \frac{1}{2^n n!} \frac{d^n}{d\mu^n} [(\mu^2 - 1)^n] \right\}$  are the associated Legendre functions, with the term in the braces the Legendre polynomials. The value  $m$  is the zonal wavenumber and the value  $n$  is obtained by adding to  $|m|$  the number of meridional nodal lines, thus  $-\infty < m < \infty$  and  $n \geq |m|$ . The term  $\sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}}$  is a normalization factor and can be divided between the Fourier and associated Legendre components such that

$$\bar{\Phi}_m(\lambda) = \frac{1}{\sqrt{2\pi}} \Phi_m(\lambda) \tag{42}$$

and

$$\bar{P}_n^m(\mu) = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}} P_n^m(\mu). \tag{43}$$

This normalization for the associated Legendre functions is used by Belousov (1962). With these definitions, the following orthonormality conditions hold:

$$\int_{-1}^1 \int_0^{2\pi} [Y_n^m(\lambda, \mu)] [Y_{n'}^{m'}(\lambda, \mu)]^* d\lambda d\mu = \delta_{mm'} \delta_{nn'}$$

$$\int_0^{2\pi} [\Phi_m(\lambda)] [\Phi_{m'}(\lambda)]^* d\lambda = \delta_{mm'}$$

$$\int_{-1}^1 \bar{P}_n^m(\mu) \bar{P}_{n'}^m(\mu) d\mu = \delta_{nn'}$$

Using these normalizations, the Fourier transform pair and the associated Legendre transform pair are, respectively,

$$\omega_{\ell m}(\mu) = \int_0^{2\pi} \omega_{\ell}(\lambda, \mu) [\Phi_m(\lambda)]^* d\lambda \tag{44a}$$

$$\omega_{\ell}(\lambda, \mu) = \sum_{m=-\infty}^{\infty} \omega_{\ell m}(\mu) \Phi_m(\lambda) \tag{44b}$$

and

$$\omega_{\ell mn} = \int_{-1}^1 \omega_{\ell m}(\mu) \bar{P}_n^m(\mu) d\mu \tag{45a}$$

$$\omega_{\ell m}(\mu) = \sum_{n=|m|}^{\infty} \omega_{\ell mn} \bar{P}_n^m(\mu). \tag{45b}$$

Because  $\omega_{\ell}(\lambda, \mu)$  is real valued, the spherical harmonic coefficients for  $m < 0$  are the complex conjugates of the spherical harmonic coefficients for  $m > 0$ . Thus, it is necessary to consider the Fourier and associated Legendre transformations only for  $m \geq 0$ .

With the notation and normalizations defined, the derivation continues with the application of the forward Fourier transform (44a) to the vertically transformed omega equation (41). The result is

$$\frac{d}{d\mu} \left[ (1-\mu^2) \frac{d\omega_{\ell m}}{d\mu} \right] - \left( \epsilon_{\ell} \mu^2 + \frac{m^2}{1-\mu^2} \right) \omega_{\ell m} = \left[ \frac{a(p_B - p_T)}{c_{\ell}} \right]^2 F_{\ell m}. \tag{46}$$

Applying the forward associated Legendre transform (45a) to (46) gives

$$n(n+1)\omega_{\ell mn} + \epsilon_{\ell} \int_{-1}^1 \omega_{\ell m}(\mu) \mu^2 \bar{P}_n^m(\mu) d\mu = - \left[ \frac{a(p_B - p_T)}{c_{\ell}} \right]^2 F_{\ell mn}, \tag{47}$$

where use is made of the fact that the normalized associated Legendre functions (as well as the associated Legendre functions) are solutions to the differential equation

$$\frac{d}{d\mu} \left[ (1-\mu^2) \frac{d\bar{P}_n^m(\mu)}{d\mu} \right] + \left[ n(n+1) - \frac{m^2}{1-\mu^2} \right] \bar{P}_n^m(\mu) = 0$$

over the interval  $-1 \leq \mu \leq 1$ . To get the integral  $\int_{-1}^1 \omega_{\ell m}(\mu) \mu^2 \bar{P}_n^m(\mu) d\mu$  into a more workable form, use the following recurrence relation for the normalized associated Legendre functions (Belousov, 1962):

$$\bar{P}_n^m(\mu) = 2a_n^m \mu \bar{P}_{n-1}^m(\mu) - b_n^m \bar{P}_{n-2}^m(\mu), \tag{48}$$

where

$$a_n^m = \sqrt{\frac{n^2 - \frac{1}{4}}{n^2 - m^2}} \quad \text{and} \quad b_n^m = \sqrt{\frac{(2n+1)(n-m-1)(n+m-1)}{(2n-3)(n^2 - m^2)}}.$$

By substituting  $n + 1$  for  $n$ , equation (48) can be written as

$$\mu \bar{P}_n^m(\mu) = \left( \frac{1}{2a_{n+1}^m} \right) \bar{P}_{n+1}^m(\mu) + \left( \frac{b_{n+1}^m}{2a_{n+1}^m} \right) \bar{P}_{n-1}^m(\mu). \tag{49}$$

Multiplying by  $\mu$  gives

$$\mu^2 \bar{P}_n^m(\mu) = \left(\frac{1}{2a_{n+1}^m}\right) \mu \bar{P}_{n+1}^m(\mu) + \left(\frac{b_{n+1}^m}{2a_{n+1}^m}\right) \mu \bar{P}_{n-1}^m(\mu). \tag{50}$$

Using (49) with  $n$  first replaced by  $n + 1$  and then with  $n$  replaced by  $n - 1$ , the factor of  $\mu$  outside the normalized associated Legendre functions may be eliminated from the right-hand side of (50), giving

$$\mu^2 \bar{P}_n^m(\mu) = \left(\frac{1}{2a_{n+1}^m}\right) \left[ \frac{1}{2a_{n+2}^m} \bar{P}_{n+2}^m(\mu) + \frac{b_{n+2}^m}{2a_{n+2}^m} \bar{P}_n^m(\mu) \right] + \left(\frac{b_{n+1}^m}{2a_{n+1}^m}\right) \left[ \frac{1}{2a_n^m} \bar{P}_n^m(\mu) + \frac{b_n^m}{2a_n^m} \bar{P}_{n-2}^m(\mu) \right].$$

Combining the  $\bar{P}_n^m(\mu)$  terms gives

$$\mu^2 \bar{P}_n^m(\mu) = \left(\frac{b_n^m b_{n+1}^m}{4a_n^m a_{n+1}^m}\right) \bar{P}_{n-2}^m(\mu) + \left(\frac{a_n^m b_{n+2}^m + a_{n+2}^m b_{n+1}^m}{4a_n^m a_{n+1}^m a_{n+2}^m}\right) \bar{P}_n^m(\mu) + \left(\frac{1}{4a_{n+1}^m a_{n+2}^m}\right) \bar{P}_{n+2}^m(\mu). \tag{51}$$

Mathematica, developed by Wolfram Research, can be used to simplify complicated coefficients such as those on the right-hand side of (51). It found that with the introduction of three new variables,  $A_{mn}$ ,  $B_{mn}$ , and  $C_{mn}$ , the coefficients can be greatly simplified. Defining  $A_{mn} = n^2 - m^2$ ,  $B_{mn} = 4n^2 - 1$ , and  $C_{mn} = \left(\frac{A_{mn}}{B_{mn}}\right)^{\frac{1}{2}}$  gives

$$\frac{b_n^m b_{n+1}^m}{4a_n^m a_{n+1}^m} = C_{mn} C_{mn-1} = \alpha_{mn},$$

$$\frac{a_n^m b_{n+2}^m + a_{n+2}^m b_{n+1}^m}{4a_n^m a_{n+1}^m a_{n+2}^m} = C_{mn+1}^2 + C_{mn}^2 = \beta_{mn},$$

and

$$\frac{1}{4a_{n+1}^m a_{n+2}^m} = C_{mn+2} C_{mn+1} = \gamma_{mn}.$$

Equation (51) can now be written as

$$\mu^2 \bar{P}_n^m(\mu) = \alpha_{mn} \bar{P}_{n-2}^m(\mu) + \beta_{mn} \bar{P}_n^m(\mu) + \gamma_{mn} \bar{P}_{n+2}^m(\mu),$$

which can be used in (47) to give

$$n(n+1)\omega_{\ell mn} + \epsilon_\ell \int_{-1}^1 \omega_{\ell m}(\mu) [\alpha_{mn} \bar{P}_{n-2}^m(\mu) + \beta_{mn} \bar{P}_n^m(\mu) + \gamma_{mn} \bar{P}_{n+2}^m(\mu)] d\mu = - \left[ \frac{a(p_B - p_T)}{c_\ell} \right]^2 F_{\ell mn}.$$

Using (45a), the above relation can be written in the form

$$A_{\ell,m,n} \omega_{\ell,m,n-2} + B_{\ell,m,n} \omega_{\ell,m,n} + C_{\ell,m,n} \omega_{\ell,m,n+2} = - \left[ \frac{a(p_B - p_T)}{c_\ell} \right]^2 F_{\ell,m,n}, \tag{52}$$

where  $A_{\ell,m,n} = \epsilon_\ell \alpha_{mn}$ ,  $B_{\ell,m,n} = n(n+1) + \epsilon_\ell \beta_{mn}$ , and  $C_{\ell,m,n} = \epsilon_\ell \gamma_{mn}$ .

According to (52), the spherical harmonic coefficients for each vertical mode  $\omega_{\ell,m,n}$  are related to the spherical harmonic coefficients of the corresponding vertical mode of the forcing  $F_{\ell,m,n}$ , and are also coupled to the spherical harmonic coefficients  $\omega_{\ell,m,n+2}$  and  $\omega_{\ell,m,n-2}$ , resulting in a system of equations for each vertical mode  $\ell$  and each zonal mode  $m$ . The indices  $m$  and  $n$  take on the respective values of  $0 \leq m < \infty$  and  $n \geq m$ . For computational purposes, however, an upper limit to the spherical harmonic coefficients must be defined. Here triangular truncation at  $n = N$  is used. For the operations related to spherical harmonics, the software package SPHEREPACK was used (Adams & Swarztrauber, 1999).

For a given vertical mode  $\ell$  and zonal mode  $m$ , the system of equations for obtaining the spherical harmonic coefficients of the global omega equation is given by:

$$\begin{bmatrix}
 B_{\ell,m,m} & 0 & C_{\ell,m,m} & 0 & 0 & 0 & \cdots & 0 \\
 0 & B_{\ell,m,m+1} & 0 & C_{\ell,m,m+1} & 0 & 0 & \cdots & 0 \\
 A_{\ell,m,m+2} & 0 & B_{\ell,m,m+2} & 0 & C_{\ell,m,m+2} & 0 & \cdots & 0 \\
 \vdots & \vdots \\
 0 & \cdots & 0 & A_{\ell,m,N-2} & 0 & B_{\ell,m,N-2} & 0 & C_{\ell,m,N-2} \\
 0 & \cdots & 0 & 0 & A_{\ell,m,N-1} & 0 & B_{\ell,m,N-1} & 0 \\
 0 & \cdots & 0 & 0 & 0 & A_{\ell,m,N} & 0 & B_{\ell,m,N}
 \end{bmatrix}
 \begin{bmatrix}
 \omega_{\ell,m,m} \\
 \omega_{\ell,m,m+1} \\
 \omega_{\ell,m,m+2} \\
 \vdots \\
 \omega_{\ell,m,N-2} \\
 \omega_{\ell,m,N-1} \\
 \omega_{\ell,m,N}
 \end{bmatrix}
 = - \left[ \frac{a(p_B - p_T)}{c_\ell} \right]^2
 \begin{bmatrix}
 F_{\ell,m,m} \\
 F_{\ell,m,m+1} \\
 F_{\ell,m,m+2} \\
 \vdots \\
 F_{\ell,m,N-2} \\
 F_{\ell,m,N-1} \\
 F_{\ell,m,N}
 \end{bmatrix}.
 \tag{53}$$

Note that the number of equations in the system depends on  $m$ . As  $m$  increases from 0 to  $N$ , the number of equations in the system decreases from  $N + 1$  to 1.

The reconstruction of  $\omega(\lambda, \mu, p)$  from the spherical harmonic coefficients  $\omega_{\ell,m,n}$  is a three-step process. First, the inverse associated Legendre transform (45b) is used to compute  $\omega_{\ell m}(\mu)$ . Next, the inverse Fourier transform (44b) is used to get  $\omega_\ell(\lambda, \mu)$ . Finally,  $\omega_\ell(\lambda, \mu)$  is summed over  $\ell$  according to the inverse vertical normal mode transfer (29b) to achieve the solution  $\omega(\lambda, \mu, p)$ .

### 3.4. The Inclusion of Diabatic Heating

The above derivation assumed an adiabatic atmosphere, an assumption commonly made for midlatitude applications for which the vorticity and thermal advection terms are relatively large. In the Tropics, the contribution to the forcing of  $\omega$  due to diabatic heating, primarily in the form of latent heat, can exceed the contributions attributable to the vorticity and thermal advection terms. To include diabatic processes as an additional forcing for large-scale vertical motion, the thermodynamic equation (8) is altered to

$$\frac{DT}{Dt} - \frac{\bar{\Gamma}}{p} \omega = \frac{J}{c_p},$$

where  $J$  represents the diabatic heating and has units of  $\frac{J}{kg s}$ . Bringing  $\frac{J}{c_p}$  through the derivation of the global omega equation, it is seen that the forcing for vertical motion due to diabatic heating enters as a Laplacian:

$$\frac{R\bar{\Gamma}}{p^2} \nabla^2 \omega + f^2 \frac{\partial^2 \omega}{\partial p^2} = -2\nabla \cdot \mathbf{Q} - f\nabla \cdot (\mathbf{k} \times \nabla \alpha) - \frac{\partial}{\partial p} (f\mathbf{k} \cdot \nabla \times \mathbf{F}) - \frac{R}{c_p p} \nabla^2 J.
 \tag{54}$$

In this case the  $\mathbf{Q}$  form of the global omega equation is shown, but the diabatic heating term appears in the same manner regardless of which omega equation is used. To include the effects of resolved-scale latent heating, one form that  $J$  might take is given by Emanuel et al. (1987) in their study of slantwise moist convection:

$$J = \omega \left( \frac{\partial \theta}{\partial p} - \frac{\gamma_m}{\gamma_d} \frac{\theta}{\theta_e} \frac{\partial \theta_e}{\partial \theta} \right),
 \tag{55}$$

where  $\theta$  is the potential temperature,  $\theta_e$  is the equivalent potential temperature,  $\gamma_d$  is the dry adiabatic lapse rate, and  $\gamma_m$  is the moist adiabatic lapse rate. Crandall et al. (2016) included this form of latent heating to

derive a diabatic  $\mathbf{Q}$ . They then compared and contrasted the diabatic  $\mathbf{Q}$  with the traditional  $\mathbf{Q}$  in the context of a numerical simulation of a cold season extratropical cyclone. It was found that the difference in the forcing due to the convergence of the traditional  $\mathbf{Q}$  and the convergence of the diabatic  $\mathbf{Q}$  was well correlated with regions of heavier modeled precipitation.

The presence of  $\omega$  in (55) means that with the inclusion of the latent heating term,  $\omega$  is both the variable for which equation (54) is solved as well as a part of the forcing. The typical manner for solving such equations is to use an iterative technique. Such is the approach given by Krishnamurti (1968).

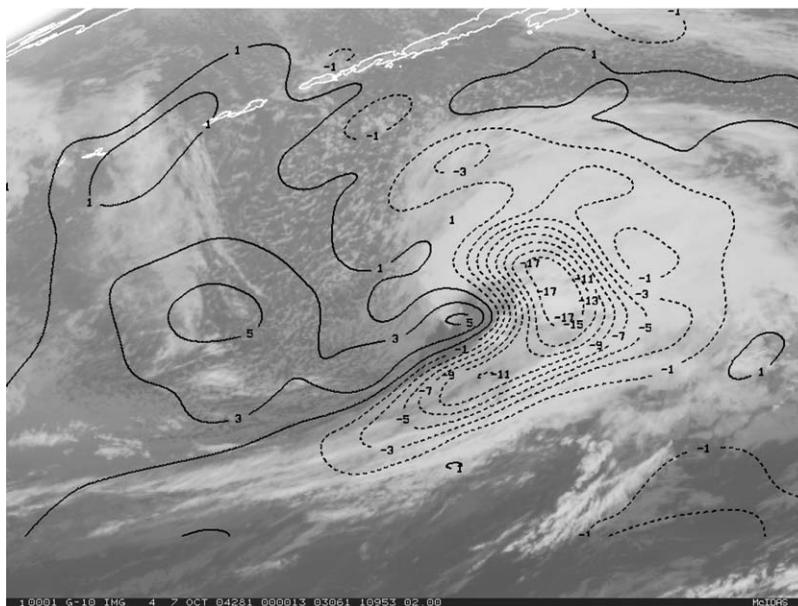
### 3.5. Two Examples

Two examples of the use of the global omega equation are given in this section. The first is an extratropical cyclone over the northeast Pacific Ocean, and the second is an African easterly wave over the eastern Atlantic Ocean. In both cases, the solution to the global omega equation is compared to the solution of the quasi-geostrophic omega equation as well as to  $\omega$  from the Global Forecast System (GFS) analysis.

#### 3.5.1. Midlatitude Cyclone

The quasi-geostrophic omega equation is used extensively in the analysis of extratropical cyclones, and applying the global omega equation to such a system provides a means by which the performance of the technique can be judged. Martin (2006) examined two extratropical cyclones using an  $f$ -plane version of the quasi-geostrophic omega equation. One of the cases was a rapidly deepening cyclone which occurred in the northeast Pacific Ocean on 6–8 October 2004. The 700-hPa omega from the 0000 UTC 7 October GFS analysis shows a pronounced couplet of rising and sinking motion (Figure 1). The upward vertical velocities peak at  $-17 \text{ dPa s}^{-1}$  in the southwestern portion of the comma head, with an extension of lifting motion to the southwest along the cold front, in agreement with the cloud shield observed by geostationary satellite. Sinking motion is present to the west of the cloud shield with two maxima exceeding  $5 \text{ dPa s}^{-1}$ . The larger of the two is well within the cold air behind the front, as evidenced by the existence of a large stratocumulus cloud field. The second maximum, at the tip of a protrusion of  $3 \text{ dPa s}^{-1}$ -subsiding air extending to the northeast, is associated with the dry tongue, or dry slot, of the cyclone's circulating airstreams (Carlson, 1980).

Using the 0000 UTC 7 October 2004 GFS analysis on a  $1^\circ \times 1^\circ$  horizontal grid with 21 vertical levels from 1,000 to 100 hPa, the field of the global  $\omega$  was calculated for the cyclone using the divergence of  $\mathbf{Q}$  (Figure 2). Similar to the GFS vertical velocity, the global  $\omega$  at 700 hPa depicts a rising/sinking couplet which



**Figure 1.** The 700-hPa  $\omega$  from the 0000 UTC 7 October 2004 GFS analysis, overlaid on a  $10.7 \mu\text{m}$  image from Geostationary Operational Environmental Satellite-10. The field is contoured every  $2 \text{ dPa s}^{-1}$ , starting at 1 and  $-1 \text{ dPa s}^{-1}$ , with upward motions dashed and subsiding motions solid.

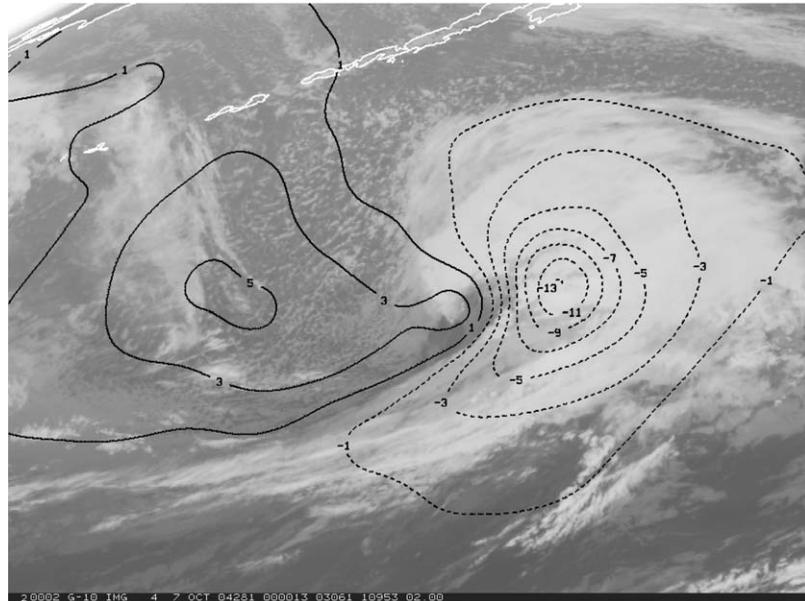


Figure 2. Same as Figure 1, but for the 700-hPa global  $\omega$ .

is consistent with the cloud fields observed from satellite. In particular, the subsiding air seems to be particularly well captured by the global omega equation, with a maximum sinking motion in the cold air mass, and an extension of the subsidence field into the dry slot, although the smaller maximum near the end of the dry-slot airstream is not evident. The maximum upward omega is coincident with that of the GFS analysis, but somewhat weaker ( $-13 \text{ dPa s}^{-1}$ ). An enhancement of  $\omega$  is evident along the southern tail of the ascent region (i.e., the cold front), and is consistent with both enhanced values of the forcing  $-2\nabla \cdot \mathbf{Q}$  along a front and with reduced values of the coefficient  $f^2$  on the south side of the cyclone. It is not surprising, however, that the upward motion near the cold front is not as narrow as in the GFS, as it has long been known that an accurate depiction of fronts requires sophistication beyond the quasi-geostrophic equations, such as semi-geostrophic theory (Hoskins, 1975).

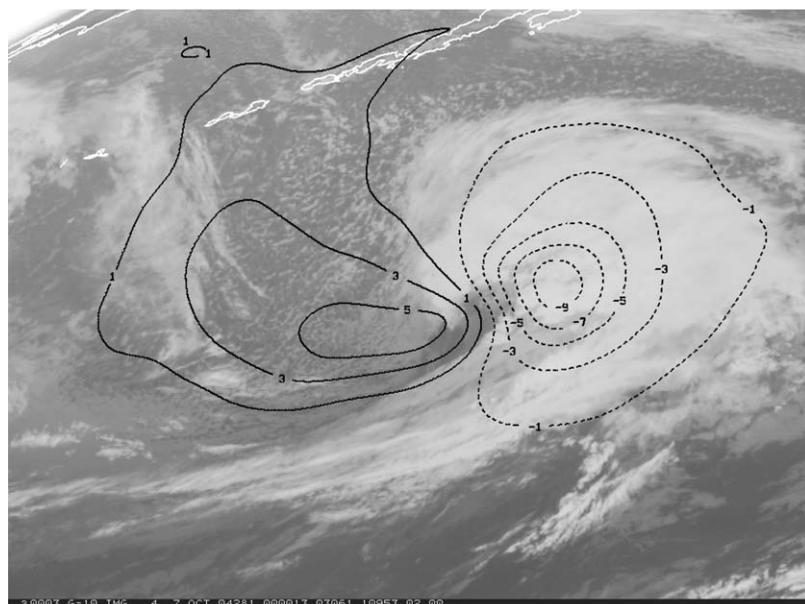
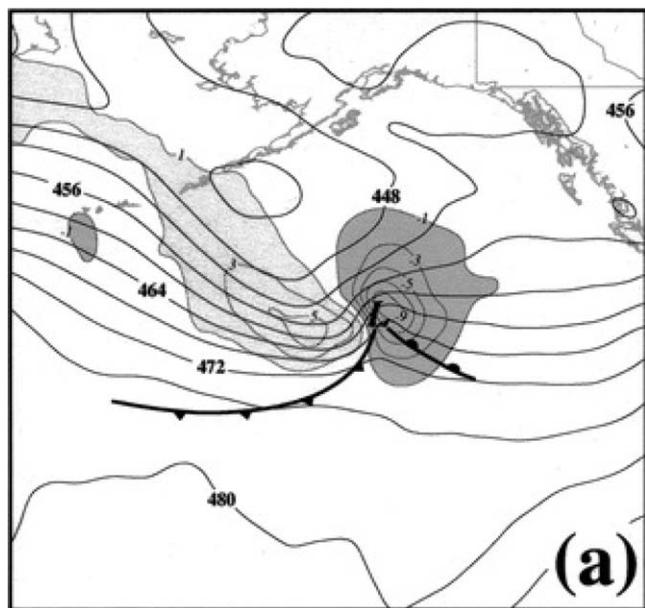


Figure 3. Same as Figure 1, but for the 700-hPa quasi-geostrophic  $\omega$ .

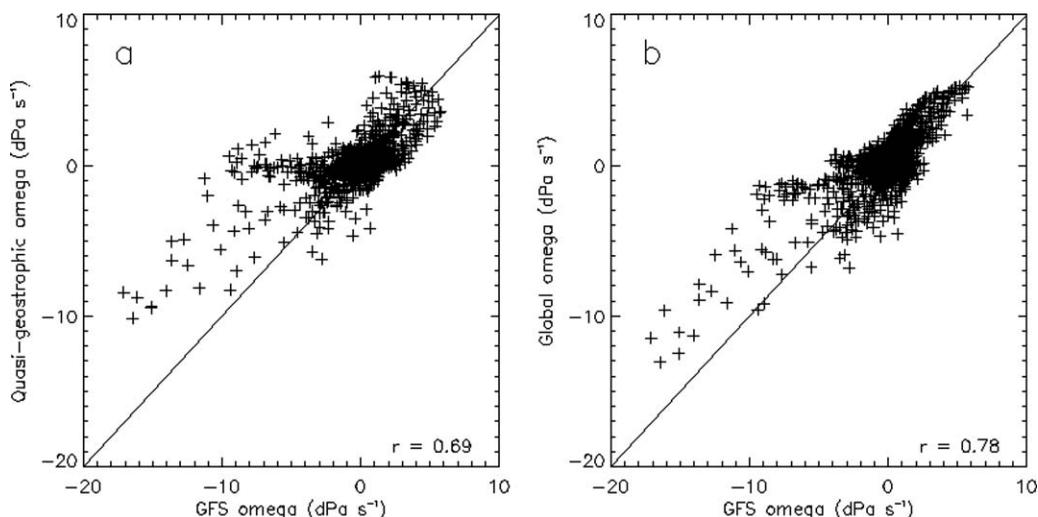


**Figure 4.** The 700-hPa  $\omega$  calculated from the quasi-geostrophic equations using successive overrelaxation, from Martin (2006). The “L” marks the position of the surface low, and the fronts are drawn according to convention. Omega is contoured every 2 dPa s<sup>-1</sup>, starting at 1 and -1 dPa s<sup>-1</sup>, with upward motions shaded in dark gray and downward motion shaded in light gray. The unshaded solid lines are the 900–500 hPa thickness, contoured every 4 dam. Courtesy of the American Meteorological Society. Copyright 2006.

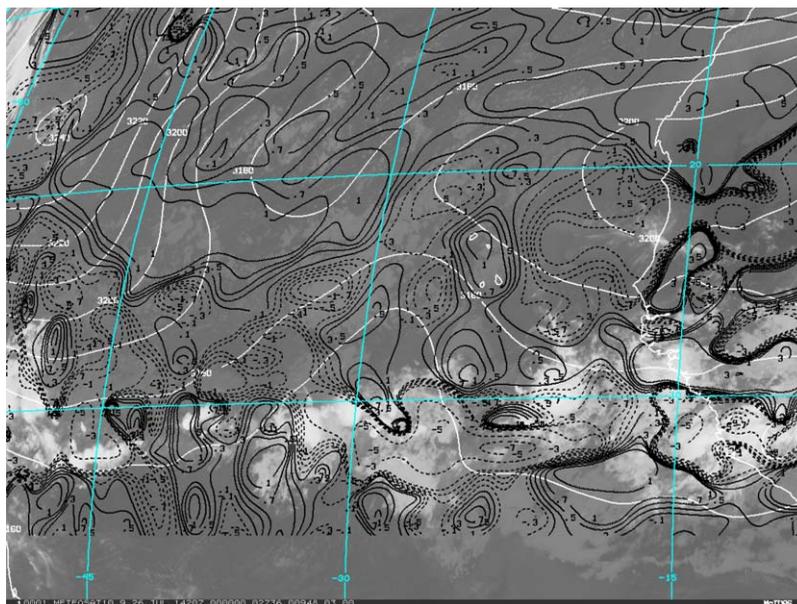
As an additional comparison, the quasi-geostrophic omega equation was also solved over the cyclone (Figure 3). For this calculation, the same GFS analysis fields were used as for the global omega equation and the display of the GFS vertical velocity, but the omega equation was solved over a limited domain surrounding the cyclone. In this case, the multigrid software package MUDPACK (Adams et al., 1992) was used to solve the partial differential equation. The boundary conditions on the edges of the domain, both lateral and vertical, were set to  $\omega$  from the GFS analysis. As with the GFS  $\omega$  and the global  $\omega$ , the quasi-geostrophic  $\omega$  displays a rising/sinking couplet of vertical motion. The magnitudes are somewhat less, with a maximum upward velocity of  $-9 \text{ dPa s}^{-1}$ , just above half the GFS model value. The location of this maximum does coincide with the locations of the maximum upward velocity of the other two fields. Similar to the global  $\omega$ , the magnitude of the largest quasi-geostrophic  $\omega$  in the sinking region matches that of the GFS analysis, but is shifted toward the cold front and the center of the cyclone, and with a much less pronounced dry slot.

Martin (2006) computed a similar balanced vertical velocity field using successive overrelaxation to solve the quasi-geostrophic omega equation from the GFS output (Figure 4). As would be expected, the strength of the rising ( $-9 \text{ dPa s}^{-1}$ ) and sinking ( $5 \text{ dPa s}^{-1}$ ) air correspond most closely to the quasi-geostrophic solution. The shape of the vertical velocity field best matches the quasi-geostrophic  $\omega$  as well, including a less pronounced dry slot than in the global or GFS  $\omega$ .

Scatter plots (Figure 5) of  $\omega$  help to quantify the performance of the two balanced models with respect to the GFS over the domain used for the midlatitude cyclone (179°W to 125°W and 35°N to 65°N). The quasi-geostrophic  $\omega$  and the global  $\omega$  both compare favorably to the GFS-analyzed  $\omega$ , although both underestimated the strongest upward motions determined by the GFS. Overall, however,  $\omega$  from the global balance model did match the GFS better—with a correlation coefficient of 0.78 compared to the quasi-geostrophic omega equation’s correlation coefficient of 0.69. The high correlation



**Figure 5.** Scatter plots of 700-hPa  $\omega$  for the extratropical cyclone of 0000 UTC 7 October 2004 in the domain 179°W to 125°W and 35°N to 65°N, excluding boundary points. Plot (a) compares the quasi-geostrophic omega equation with the GFS analysis, and plot (b) compares the global omega equation with the GFS analysis. The resulting correlation coefficient is given in the lower right-hand corner of each plot.



**Figure 6.** The 700-hPa height and  $\omega$  as analyzed by the GFS for 0000 UTC 26 July 2014, overlaid on a Meteosat-10 10.8  $\mu\text{m}$  image. Heights are contoured every 10 m and vertical motions contoured every  $0.2 \text{ dPa s}^{-1}$ , starting at  $0.1$  and  $-0.1 \text{ dPa s}^{-1}$ , to  $0.7$  and  $-0.7 \text{ dPa s}^{-1}$ , and then every  $2 \text{ dPa s}^{-1}$  starting at  $1$  and  $-1 \text{ dPa s}^{-1}$ . Solid contours indicate sinking motion and dashed contours indicate rising motion.

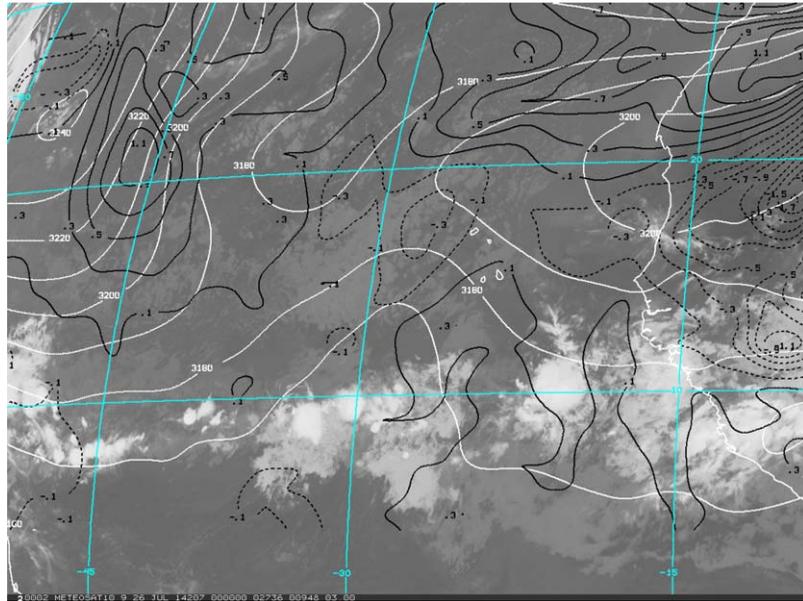
coefficients produced by both balanced models underscore the importance of adiabatic forcing on the behavior of midlatitude systems.

### 3.5.2. Tropical Wave

The easterly wave that would eventually develop into Hurricane Bertha of the 2014 Atlantic hurricane season left the west coast of Africa on 24 July 2014, and by 26 July it was over the ocean and displayed the classic “inverted V” pattern in the GFS 700-hPa height field (Figure 6). The accompanying vertical motion field reveals much more fine-scale structure than in the midlatitude example discussed previously. This feature is expected, as vertical velocity at low latitudes is much more dependent on the diabatic heating associated with convection, a process which occurs on a smaller scale than the forcing associated with the divergence of  $\mathbf{Q}$ . The small scale is particularly pronounced in the convective cloud clusters of the Intertropical Convergence Zone (ITCZ) to the south of the wave, as seen in the satellite imagery. A pattern of vertical velocity within the tropical wave is nevertheless evident, with generally rising air in the western portion of the wave and sinking motion to the east, in agreement with Baumhefner (1968), who noted that pattern when examining an easterly wave in the Caribbean Sea using an omega equation based on the nonlinear balance model of Krishnamurti (1968). The crest of the wave contains regions of both sinking and rising motion, the latter extending north from the wave crest into the western extent of the ridge to the north.

Due to the absence of diabatic heating, the field of vertical velocity associated with the global omega equation for this case (Figure 7) lacks the fine-scale structure of the GFS  $\omega$ . The large-scale rising and sinking motions in the vicinity of the wave are consistent with the numerical model output. The rising on the western portion of the wave is less pronounced, however, and the overall magnitudes of both the rising and sinking regions are smaller as well. The global  $\omega$  shows only rising motion in the wave crest, but is consistent with the GFS analysis in that the upward motions extend into the western portions of the ridge to the north.

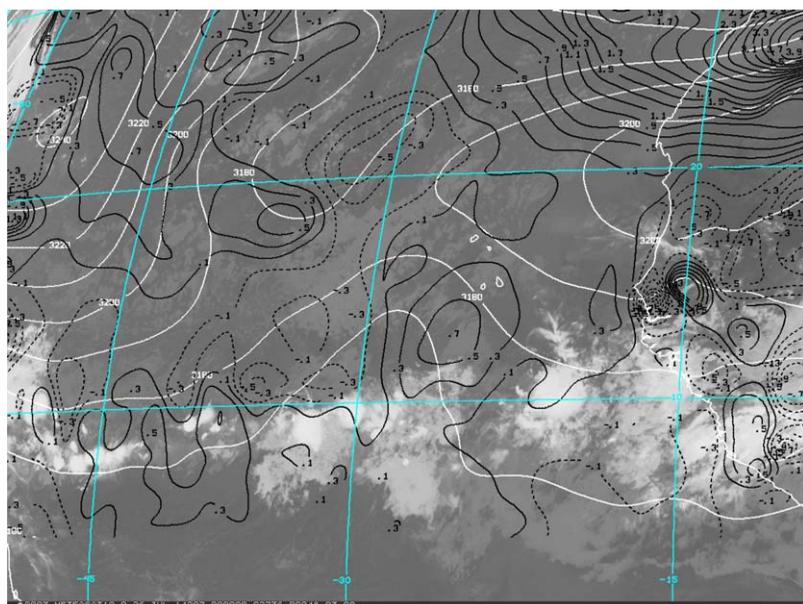
As with the global  $\omega$ , the quasi-geostrophic  $\omega$  (Figure 8) is dominated by large-scale features, such as the subsidence on the eastern portion of the wave. The western portion of the wave is dominated by upward motion, which is actually more pronounced than that of the global omega equation. The crest of the wave serves as a partition between these rising and sinking motions, with only a slight hint of the upward



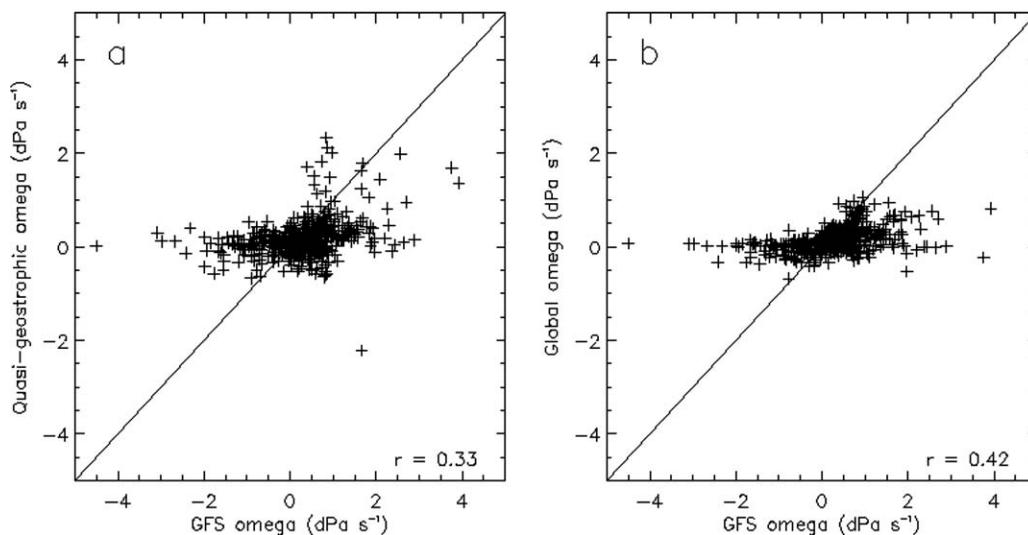
**Figure 7.** As in Figure 6, but with  $\omega$  given by the global omega equation. The contour interval of  $\omega$  is every  $0.2 \text{ dPa s}^{-1}$  starting at  $0.1$  and  $-0.1 \text{ dPa s}^{-1}$ .

motions extending into the ridge to the north. The magnitudes of both the rising and sinking motions are similar to the global  $\omega$ , that is, generally less than those calculated by the GFS model. Note also that the magnitudes of  $\omega$  from both balanced systems in the Tropics are 1–2 orders of magnitude smaller than the magnitudes of omega in the midlatitude case, again underscoring the importance of adiabatic motions in the extratropics and diabatic processes at lower latitudes.

The overall performance of the two balanced omega equations in the region of the tropical wave and its environment can be more quantitatively analyzed by again using scatter plots to compare the vertical motions at 700 hPa with the GFS analysis. Compared to the extratropical cyclone, the tropical wave



**Figure 8.** As Figure 7, but with  $\omega$  given by the quasi-geostrophic omega equation.



**Figure 9.** As in Figure 5, but for the tropical wave of 0000 UTC 26 July 2014, with an analysis domain of 10°N to 25°N and 45°W to 15°W.

occupies a smaller fraction of the domain over which the quasi-geostrophic omega was calculated (1°N to 31°N and 55°W to 1°W). In order to focus on the tropical wave, the domain used for the scatter plots was limited to 10°–25°N and 15°–45°W. Using 10°N as the southern boundary removes from consideration most of the grid points dominated by convection along the ITCZ, which the global and quasi-geostrophic omega equations will not be able to capture because their forcing is restricted to adiabatic processes.

The results are displayed in Figure 9. Immediately clear is the overall poorer fit between the balanced  $\omega$  fields and the GFS values than in the midlatitude case. In particular, the global and quasi-geostrophic systems have difficulty replicating the GFS analysis for larger magnitudes of  $\omega$  ( $\geq 1$  dPa s<sup>-1</sup> or  $\leq -1$  dPa s<sup>-1</sup>). Vertical motions in these ranges likely correspond to updrafts and downdrafts produced by convection that remained after the simplistic method to effect its removal was applied. For magnitudes of  $\omega$  less than 1 dPa s<sup>-1</sup>, the fit is better visually, and more of the global  $\omega$  values appear to lie along the 1:1 line than the  $\omega$  values produced by the quasi-geostrophic equation. This qualitative evaluation is confirmed by the computation of the correlation coefficients, 0.42 for the global  $\omega$  equation and 0.33 for the quasi-geostrophic  $\omega$  equation.

#### 4. Conclusions

A balanced system of equations, including its associated omega equation, has been derived for the entire sphere. In the derivation, three key steps are taken. First, the horizontal wind is decomposed into a nondivergent and an irrotational component instead of the geostrophic and ageostrophic components as used in quasi-geostrophic theory. Second, the Coriolis parameter is assumed to be slowly varying with respect to the latitudinal variation of the streamfunction defining the nondivergent flow. This assumption leads to a balance between the geopotential and the streamfunction given by  $f\psi(\lambda, \mu, p) = \Phi(\lambda, \mu, p) - \bar{\Phi}(p)$ . Third, when creating the atmosphere which contains a balance between the wind and mass fields, it is necessary to begin by defining the streamfunction of the wind, and then compute the mass field in balance with it. From the balance condition, it can be seen that this method is the only way to avoid a discontinuity at the equator. Furthermore, Phillips (1958) found in the geostrophic context that it is more accurate to begin with a wind field and from there compute the balanced mass field, than to begin with a mass field and from it compute the balanced wind field. With these three steps, the derivation of a balanced system of equations valid for the whole sphere proceeds in a manner similar to that used to derive the quasi-geostrophic set of equations. Not surprisingly, the resulting set of equations is essentially the same in appearance as the quasi-geostrophic equations.

From this new set of equations, the derivation of the global omega equation again follows just as in the quasi-geostrophic case. The equation takes the form

$$\frac{R\bar{\Gamma}}{p^2} \nabla^2 \omega + f^2 \frac{\partial^2 \omega}{\partial p^2} = \frac{\partial}{\partial p} [f \mathbf{v}_\psi \cdot \nabla (\zeta + f)] - \nabla^2 \left[ f \mathbf{v}_\psi \cdot \nabla \left( \frac{\partial \psi}{\partial p} \right) \right] - \frac{\partial}{\partial p} (f \mathbf{k} \cdot \nabla \times \mathbf{F}).$$

That is, the large-scale vertical motion associated with this balanced system of equations is forced by the vertical derivative of vorticity advection by the nondivergent wind and the Laplacian of the advection of the temperature by the nondivergent wind. It is similar to the quasi-geostrophic omega equation except that horizontal advectons are accomplished by the nondivergent wind instead of the geostrophic wind, and that the Coriolis parameter is allowed its full variation. This variation is assumed to be small with respect to the variation of the streamfunction however, meaning that the approximation breaks down as the meridional variations of the streamfunction decrease (sectoral harmonics). In addition to this "traditional" form of the global omega equation, a **Q** form of the global omega equation was derived, where **Q** takes a form similar to that associated with the quasi-geostrophic omega equation.

In order to solve the global omega equation, a normal mode transform in the vertical and a spherical harmonic transform in the horizontal were employed. This transformation results in a system of equations for each vertical and zonal mode which is then solved to get the spherical harmonic coefficients. The inverse Legendre, Fourier, and vertical normal mode transforms are then performed to arrive at the solution in physical space.

Two examples, an extratropical cyclone and an African easterly wave, provide a comparison among the global omega equation, the quasi-geostrophic omega equation, and  $\omega$  computed by the GFS numerical model. The comparisons demonstrate the well-known dominance of adiabatic forcing of vertical motion in the midlatitudes and diabatic forcing in the Tropics. In the midlatitude case, the global omega equation produced a vertical velocity field more consistent with the numerical model output than the quasi-geostrophic omega equation, both by inspection of the vertical motion fields and by the generation of scatter plots and the computation of the associated correlation coefficients. In the tropical wave case, visual inspection was more ambiguous, as the vertical motion field of the GFS model was dominated by the small-scale effects of convection. The quantitative assessment by the scatterplots and the correlation coefficients did, however, suggest improved performance with the global omega equation over the quasi-geostrophic omega equation.

The results presented in this work point to the global omega equation as broadly applicable and more accurate than the quasi-geostrophic omega equation. Because it is valid over the entire globe, it can be used to examine the vertical velocities forced by balanced flow at all latitudes (e.g., midlatitude and tropical cyclones), as well as systems covering large portions of the globe and are not dominated by sectoral harmonics (e.g., jet streams). Only in the extratropics, however, can it be considered a reasonable proxy for the actual omega, as it is in these regions that adiabatic motions typically dominate the forcing of the vertical motion field.

### Appendix A:: Orthonormality of $\frac{dV_\ell}{dp}$ and the Vertical Transform Pair for $\omega$

In section 3, the equations governing the orthonormality of  $\frac{dV_\ell}{dp}$  (34) and the transform pair for  $\omega$ , (29a) and (29b), were presented without derivation. In this appendix, each equation will be derived, starting with the orthonormality condition on  $\frac{dV_\ell}{dp}$ .

#### A1. Orthonormality Condition on $\frac{dV_\ell}{dp}$

Fulton and Schubert (1985) demonstrate the orthonormality condition (33) obeyed by the structure function  $V_\ell(p)$ , which is repeated here:

$$\frac{1}{p_B - p_T} \int_{p_T}^{p_B} V_\ell V_{\ell'} dp = \begin{cases} 1 & \text{if } \ell' = \ell \\ 0 & \text{if } \ell' \neq \ell. \end{cases}$$

Using (30) to substitute for  $V_\ell$  gives

$$\begin{aligned} \frac{1}{p_B - p_T} \int_{p_T}^{p_B} V_\ell V_{\ell'} dp &= -\frac{c_\ell^2}{p_B - p_T} \int_{p_T}^{p_B} V_\ell \frac{d}{dp} \left( \frac{p^2 dV_\ell}{R\bar{\Gamma}} \right) dp \\ &= -\frac{c_\ell^2}{p_B - p_T} \left[ \int_{p_T}^{p_B} \frac{d}{dp} \left( V_\ell \frac{p^2 dV_\ell}{R\bar{\Gamma}} \right) dp - \int_{p_T}^{p_B} \frac{p^2 dV_\ell}{R\bar{\Gamma}} \frac{dV_\ell}{dp} dp \right] \\ &= \begin{cases} 1 & \text{if } \ell' = \ell \\ 0 & \text{if } \ell' \neq \ell. \end{cases} \end{aligned}$$

Evaluating the first integral and factoring out a minus sign gives

$$\frac{c_\ell^2}{p_B - p_T} \left[ - \left( V_\ell \frac{p^2 dV_\ell}{R\bar{\Gamma}} \right)_{p_T}^{p_B} + \int_{p_T}^{p_B} \frac{p^2 dV_\ell}{R\bar{\Gamma}} \frac{dV_\ell}{dp} dp \right] = \begin{cases} 1 & \text{if } \ell' = \ell \\ 0 & \text{if } \ell' \neq \ell. \end{cases}$$

Because  $\frac{dV_\ell}{dp} = 0$  at  $p = p_T$ , the above equation can be written as

$$\frac{c_\ell^2}{p_B - p_T} \left[ - \left( V_\ell \frac{p^2 dV_\ell}{R\bar{\Gamma}} \right)_{p_B} + \int_{p_T}^{p_B} \frac{p^2 dV_\ell}{R\bar{\Gamma}} \frac{dV_\ell}{dp} dp \right] = \begin{cases} 1 & \text{if } \ell' = \ell \\ 0 & \text{if } \ell' \neq \ell. \end{cases}$$

Using  $V_{\ell'} = -\frac{p_T}{\bar{\Gamma}} \frac{dV_{\ell'}}{dp}$  at  $p = p_B$ , the orthonormality condition on  $\frac{dV_\ell}{dp}$  is given by

$$\frac{c_\ell^2}{p_B - p_T} \left[ \int_{p_T}^{p_B} \frac{p^2 dV_\ell}{R\bar{\Gamma}} \frac{dV_{\ell'}}{dp} dp + \left( \frac{p^3 \bar{\Gamma}}{R\bar{\Gamma}^2} \frac{dV_\ell dV_{\ell'}}{dp} \right)_{p_B} \right] = \begin{cases} 1 & \text{if } \ell' = \ell \\ 0 & \text{if } \ell' \neq \ell, \end{cases}$$

which is the same as (34).

### A2. The Inverse Transform for $\omega$

In deriving the inverse transform for  $\omega$ , begin with the inverse transform of the components of the horizontal wind,  $\mathbf{v}(\lambda, \mu, p) = \sum_{\ell=0}^{\infty} \mathbf{v}_\ell(\lambda, \mu) V_\ell(p)$  (Fulton & Schubert, 1985), and use  $\omega(\lambda, \mu, p) = \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) A_\ell(p)$ , where  $A_\ell(p)$  is to be determined. Begin by inserting these equations into the continuity equation (3):

$$\nabla \cdot \sum_{\ell=0}^{\infty} \mathbf{v}_\ell(\lambda, \mu) V_\ell + \frac{\partial}{\partial p} \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) A_\ell(p) = 0$$

or

$$\sum_{\ell=0}^{\infty} [\nabla \cdot \mathbf{v}_\ell(\lambda, \mu)] V_\ell + \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) \frac{dA_\ell(p)}{dp} = 0.$$

Multiply by  $\frac{V_\ell}{p_B - p_T}$  and integrate from  $p_T$  to  $p_B$  to get

$$\sum_{\ell=0}^{\infty} [\nabla \cdot \mathbf{v}_\ell(\lambda, \mu)] \left( \frac{1}{p_B - p_T} \right) \int_{p_T}^{p_B} V_\ell V_{\ell'} dp + \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) \left( \frac{1}{p_B - p_T} \right) \int_{p_T}^{p_B} \frac{dA_\ell}{dp} V_\ell dp = 0.$$

If  $\frac{dA_\ell(p)}{dp} = -\frac{V_\ell}{p_B - p_T}$ , the equation becomes

$$\sum_{\ell=0}^{\infty} \left\{ \left[ \nabla \cdot \mathbf{v}_\ell(\lambda, \mu) - \left( \frac{1}{p_B - p_T} \right) \omega_\ell(\lambda, \mu) \right] \left( \frac{1}{p_B - p_T} \right) \int_{p_T}^{p_B} V_\ell V_{\ell'} dp \right\} = 0.$$

Using the orthonormality condition on  $V_\ell$ , the continuity equation for each vertical mode may be written in a form similar to (3), where  $\omega_\ell(\lambda, \mu)$  has the same units as  $\omega(\lambda, \mu, p)$ :

$$\nabla \cdot \mathbf{v}_\ell(\lambda, \mu) = \left( \frac{1}{p_B - p_T} \right) \omega_\ell(\lambda, \mu).$$

Looking further at  $\frac{dA_\ell(p)}{dp} = -\frac{V_\ell}{p_B - p_T}$ , a substitution for  $V_\ell$  from (30) gives

$$\frac{dA_\ell(p)}{dp} = \frac{c_\ell^2}{p_B - p_T} \frac{d}{dp} \left[ \frac{p^2 dV_\ell}{R\bar{\Gamma}} \right]$$

or

$$\frac{d}{dp} \left[ A_\ell(p) - \frac{c_\ell^2}{p_B - p_T} \frac{p^2 dV_\ell}{R\bar{\Gamma}} \right] = 0.$$

An indefinite integral over  $p$  gives

$$A_\ell(p) = \frac{c_\ell^2}{p_B - p_T} \frac{p^2 dV_\ell}{R\bar{\Gamma}} + C.$$

In order for  $\omega(\lambda, \mu, p) = \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) A_\ell(p)$  to remain bounded, it must be that  $C = 0$ . Therefore, the inverse transform for  $\omega$  is

$$\omega(\lambda, \mu, p) = \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) \frac{c_\ell^2 p^2}{R\bar{\Gamma}(p_B - p_T)} \frac{dV_\ell}{dp}, \quad (A1)$$

which is the same as (29b).

### A3. Derivation of the Forward Transform for $\omega$

To derive the forward transform of  $\omega$ , begin with the inverse transform (A1) applied at the lower boundary:

$$[\omega(\lambda, \mu, p)]_{p_B} = \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) \left[ \frac{c_\ell^2 p^2}{R\bar{\Gamma}(p_B - p_T)} \frac{dV_\ell}{dp} \right]_{p_B}. \quad (A2)$$

Now multiply (A1) by  $\frac{dV_{\ell'}}{dp}$  and integrate from  $p_T$  to  $p_B$ , and multiply (A2) by  $\left( \frac{p\bar{T}}{\bar{\Gamma}} \frac{dV_{\ell'}}{dp} \right)_{p_B}$ . Adding the two equations gives:

$$\int_{p_T}^{p_B} \omega(\lambda, \mu, p) \frac{dV_{\ell'}}{dp} dp + \left[ \omega(\lambda, \mu, p) \frac{p\bar{T}}{\bar{\Gamma}} \frac{dV_{\ell'}}{dp} \right]_{p_B} = \sum_{\ell=0}^{\infty} \omega_\ell(\lambda, \mu) \left\{ \frac{c_\ell^2}{p_B - p_T} \left[ \int_{p_T}^{p_B} \frac{p^2 dV_\ell}{R\bar{\Gamma}} \frac{dV_{\ell'}}{dp} dp + \left( \frac{p^3 \bar{T}}{R\bar{\Gamma}^2} \frac{dV_\ell}{dp} \frac{dV_{\ell'}}{dp} \right)_{p_B} \right] \right\}.$$

From the orthonormality condition on  $\frac{dV_{\ell'}}{dp}$ , the expression in the braces assumes one of two values:

$$\begin{aligned} &1 && \text{if } \ell = \ell' \\ &0 && \text{if } \ell \neq \ell'. \end{aligned}$$

Using that result, the forward transform is given by

$$\omega_\ell(\lambda, \mu) = \int_{p_T}^{p_B} \omega(\lambda, \mu, p) \frac{dV_\ell}{dp} dp + \left[ \omega(\lambda, \mu, p) \frac{p\bar{T}}{\bar{\Gamma}} \frac{dV_\ell}{dp} \right]_{p_B},$$

which is the same as (29a).

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